

A Numerical Method for SDEs with Discontinuous Drift

Gunther Leobacher and Michaela Szölgyenyi ^{*}

Preprint, January 2015

Abstract

In this paper we introduce a transformation technique, which can on the one hand be used to prove existence and uniqueness for a class of SDEs with discontinuous drift coefficient. On the other hand we present a numerical method based on transforming the Euler-Maruyama scheme for such a class of SDEs. We prove convergence of order 1/2. Finally, we present numerical examples.

Keywords: stochastic differential equations, discontinuous drift, numerical methods for stochastic differential equations

Mathematics Subject Classification (2010): 60H10, 65C30, 65C20 (Primary), 65L20 (Secondary)

G. Leobacher

Department of Financial Mathematics, Johannes Kepler University Linz, 4040 Linz, Austria

M. Szölgyenyi \boxtimes

Department of Financial Mathematics, Johannes Kepler University Linz, 4040 Linz, Austria

michaela.szoelgyenyi@jku.at

1 Introduction

We consider a time-homogeneous stochastic differential equation (SDE)

$$dX_t = \mu(X_t) dt + \sigma(X_t) dW_t, \quad (1)$$

where σ is a Lipschitz, \mathbb{R} -valued function and where μ is an \mathbb{R} -valued function that is allowed to have discontinuities.

It is well-known that the problem of existence and uniqueness of a solution to (1) is readily settled by Picard iteration, if μ is Lipschitz, too. In the case where the diffusion coefficient is bounded, Lipschitz, and (partly) uniformly elliptic, and the drift coefficient μ is only bounded and measurable, the pioneering work by Zvonkin [15] and Veretennikov [13, 14] yields existence and uniqueness of the solution. There the result is achieved by applying a transform that removes the drift. This transform can in principle be computed by solving a non-degenerate elliptic partial differential equation. In the one-dimensional case this reduces to solving an ordinary differential equation. Nevertheless, from the point of view of numerical treatment of (1) this transformation method is impractical.

In Leobacher et al. [11] one can find an existence and uniqueness result for multi-dimensional SDEs, for the case where the drift is allowed to be discontinuous at a hyperplane, or at a hypersurface, but is

^{*}The authors thank Evelyn Buckwar (Johannes Kepler University Linz) for valuable discussions. G. Leobacher and M. Szölgyenyi are supported by the Austrian Science Fund (FWF): Project F5508-N26, which is part of the Special Research Program "Quasi-Monte Carlo Methods: Theory and Applications".

well behaved everywhere else. Instead of removing the whole drift by a transformation, only the discontinuity is removed by using a different transformation method. For computing this transform one only needs to solve a parametrized family of ODEs which can be done by iterated integration. This transform is therefore much less costly from a computational point of view.

In this paper, we prove another existence and uniqueness result for (1) under conditions weaker than those in Leobacher et al. [11]. The transform constructed here is explicit and thus gives rise to a numerical method that does not require solving any (partial) differential equation.

In setups with non-globally Lipschitz drift coefficient, various authors have studied convergence of numerical schemes. Berkaoui [2] proves strong convergence of the Euler-Maruyama scheme for C^1 drift. Hutzenthaler et al. [7] present an explicit numerical method for which they are able to prove strong convergence in case of an SDE with non-globally Lipschitz coefficient. Gyöngy [5] proves almost sure convergence of the Euler-Maruyama scheme in the case where the drift satisfies a monotonicity condition. Halidias and Kloeden [6] show that the Euler-Maruyama scheme converges strongly in case of a discontinuous monotone drift coefficient, e.g., in the case where the drift is a Heaviside function. Kohatsu-Higa et al. [10] show – in case of a discontinuous drift – weak convergence of a method where they first regularize the drift and then apply the Euler-Maruyama scheme. Étoré and Martinez [3, 4] present an exact simulation algorithm for SDEs with a bounded drift coefficient that has a discontinuity in one point, but is differentiable everywhere else.

In contrast to that, we allow the drift to have a finite number of jumps, but require it to be Lipschitz otherwise. The transformation method and the scheme based on it are the main contributions of this paper. The transformation itself is presented in a constructive way. Moreover, it is chosen such that all the involved functions can be computed efficiently.

The paper is organized as follows. In Section 2 we introduce a transformation method different to those from [11, 15] and show that this also leads to an existence and uniqueness result. In Section 3 we present the numerical method which is based on the transformation introduced earlier and we prove convergence of strong order 1/2. Finally, in Section 4 we present numerical examples.

2 Existence and uniqueness

Let $(\mathcal{E}, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$ be a filtered probability space carrying a standard Brownian motion $W = (W_t)_{t \geq 0}$. Let $\mu, \sigma : \mathbb{R} \rightarrow \mathbb{R}$, be measurable functions. We study the time-homogeneous stochastic differential equation (SDE)

$$dX_t = \mu(X_t) dt + \sigma(X_t) dW_t, \quad X_0 = x. \quad (2)$$

The function μ is allowed to be discontinuous. However, the form of the discontinuities is a special one: we allow only discontinuities in a finite number m of distinct points $\xi_1 < \dots < \xi_m$ and we assume that μ is Lipschitz otherwise.

Definition 2.1. Let $I \subseteq \mathbb{R}$ be an interval. We say a function $f : I \rightarrow \mathbb{R}$ is piecewise Lipschitz if there are finitely many points $\xi_1 < \dots < \xi_m \in I$ such that f is Lipschitz on each of the intervals $(-\infty, \xi_1) \cap I$, $(\xi_m, \infty) \cap I$ and (ξ_k, ξ_{k+1}) , $k = 1, \dots, m$.

We have not assumed anything about the behaviour of f at ξ_1, \dots, ξ_m . However, we have the following elementary lemma:

Lemma 2.2. *For a piecewise Lipschitz function f the one-sided limits always exist.*

Proof. By symmetry it is enough to show that the left limit exists in every point. For given $x \in I$ consider a non-decreasing sequence (x_n) with $x_n \rightarrow x$. According to our assumption there is some L and some n_0 such that $|f(x_n) - f(x_k)| \leq L|x_n - x_k|$ for all $n, k \geq n_0$. Now

$$\sum_{n=n_0}^{\infty} |f(x_{n+1}) - f(x_n)| \leq \sum_{n=n_0}^{\infty} L|x_{n+1} - x_n| = L \sum_{n=n_0}^{\infty} (x_{n+1} - x_n) = L(x - x_{n_0}) < \infty$$

such that $\sum_{n=n_0}^{\infty} (f(x_{n+1}) - f(x_n))$ converges absolutely and the limit of this series is $\lim_{n \rightarrow \infty} f(x_n) - f(x_{n_0})$. \square

This will enable us to transform the SDE into one with Lipschitz coefficients. The transform presented here is similar to the one in [11] but much simpler to compute.

Assumption 2.3. We assume the following for the coefficients of (2):

- (i) μ is piecewise Lipschitz;
- (ii) σ is globally Lipschitz;
- (iii) There exists $\bar{c} > 0$ such that $\sigma^2(\xi_i) \geq \bar{c}$ for all $i = 1, \dots, m$.

We want to stress that those assumptions are satisfied by many practical examples. A classical one is the one-dimensional process X satisfying $X_0 = x$ and

$$dX_t = -\text{sign}(X_t)dt + dW_t.$$

Our assumptions on μ enable us to remove the discontinuity from the drift by a suitable transformation of X . The transformation g is chosen such that g'' is piecewise linear and is non-zero only on environments of the discontinuities. This yields a piecewise quadratic function g' , which is constantly 1 except on environments of the discontinuities. Furthermore, we can guarantee boundedness of the derivatives of g .

Proposition 2.4. Let $\xi_1, \dots, \xi_m \in \mathbb{R}$ with $\xi_1 < \dots < \xi_m$ and let $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m \in \mathbb{R}$. Let $0 < \varkappa < 1$ be fixed.

There exist functions $g, h : \mathbb{R} \rightarrow \mathbb{R}$ such that

1. $g(h(z)) = z$ for all $z \in \mathbb{R}$ and $h(g(x)) = x$ for all $x \in \mathbb{R}$;
2. $g, h \in C^1(\mathbb{R})$;
3. $\|g' - 1\|_{\infty} \leq \varkappa$ and $\|h' - 1\|_{\infty} \leq \varkappa$;
4. $\sup_{x \in \mathbb{R}} |g(x) - x| \leq \frac{\varkappa}{2} \max(1, \max_{1 \leq k \leq m} (\xi_{k+1} - \xi_k))$ and $g(x) = x$ for $|x|$ sufficiently large;
5. g'', h'' are piecewise continuous and bounded on \mathbb{R} , both one-sided limits of g'' at ξ_k exist, and both one-sided limits of h'' at $g(\xi_k)$ exist for $k = 1, \dots, m$;
6. $g''(\xi_k+) = \alpha_k$ and $g''(\xi_k-) = \beta_k$, $k = 1, \dots, m$.

Proof. Let $\xi_0 = \xi_1 - 1$, $\xi_{m+1} = \xi_m + 1$, $\alpha_0 = \beta_0 = \alpha_{m+1} = \beta_{m+1} = 0$. We construct g'' on the intervals $[\xi_{k-1}, \xi_k]$ for $k = 1, \dots, m+1$.

Choose $c_k \in [\xi_{k-1}, \xi_{k-1} + \frac{1}{4}(\xi_{k-1} + \xi_k))$ (c_k will depend on \varkappa) and let g'' on $[\xi_{k-1}, \frac{1}{2}(\xi_{k-1} + \xi_k))$ be

$$g''(x) = \begin{cases} \alpha - \frac{3\alpha(x-\xi)}{c-\xi}, & x \in (\xi, \frac{1}{2}(\xi + c)] \\ \alpha \frac{x-c}{c-\xi}, & x \in (\frac{1}{2}(\xi + c), c] \\ -\frac{8\alpha(x-c)}{3(c-\xi)}, & x \in (c, c + \frac{c-\xi}{4}] \\ \frac{4\alpha(\xi-c+2(x-c))}{3(c-\xi)}, & x \in (c + \frac{c-\xi}{4}, c + \frac{3(c-\xi)}{4}] \\ -\frac{8\alpha(\xi-2c+x)}{3(x-\xi)}, & x \in (c + \frac{3(c-\xi)}{4}, 2c - \xi] \\ 0, & x \in (2c - \xi, \frac{1}{2}(\xi + \xi_k)], \end{cases}$$

where we write $\xi = \xi_{k-1}$, $\alpha = \alpha_{k-1}$, and $c = c_k$ for brevity. Note that

$$\int_{\xi_{k-1}}^{\frac{\xi_{k-1}+\xi_k}{2}} g''(x) dx = 0 \quad \text{and} \quad \int_{\xi_{k-1}}^{\frac{\xi_{k-1}+\xi_k}{2}} \int_{\xi_{k-1}}^x g''(t) dt dx = 0,$$

and that we choose c_k such that

$$\max_{\xi_{k-1} \leq x \leq \frac{\xi_{k-1}+\xi_k}{2}} \left| \int_{\xi_{k-1}}^x g''(t) dt \right| \leq \varkappa/(1 + \varkappa).$$

On $[\frac{1}{2}(\xi_{k-1} + \xi_k), \xi_k]$ we define g'' analog with $g''(\xi_k) = \beta_k$. Further define $g''(x) = 0$ for $x < \xi_0$ and $x > \xi_{m+1}$.

Now let

$$g(x) = x + \int_0^x \int_0^t g''(s) ds dt.$$

Then it is easy to verify that g satisfies items 2 – 6.

The function g is piecewise cubic with positive derivative and therefore has a global inverse h that can be given explicitly as a piecewise radical function. We have

$$|h'(z) - 1| = \left| \frac{1}{g'(h(z))} - 1 \right| = \frac{|1 - g'(h(z))|}{|g'(h(z))|} \leq \frac{\varkappa}{1 - \frac{\varkappa}{1 + \varkappa}} = \varkappa.$$

□

The proof of Proposition 2.4 is constructive. So now we know how we can construct the function for the numerical approximation. Figure 1 shows the functions g' and g'' on the right of some ξ .

The function g'' is piecewise Lipschitz. The function g' , or the constants $\alpha_1, \beta_1, \dots, \alpha_m, \beta_m$ are chosen such that the discontinuities are removed from the drift in a way such that the remaining term is locally Lipschitz, i.e., $\alpha_k = 2\frac{\bar{\mu}_k - \mu(\xi_k+)}{\sigma^2(\xi_k)}$, $\beta_k = 2\frac{\bar{\mu}_k - \mu(\xi_k-)}{\sigma^2(\xi_k)}$, and $\bar{\mu}_k = \frac{\mu(\xi_k-) + \mu(\xi_k+)}{2}$. We define the transformed SDE by $Z = g(X)$. Then

$$dZ_t = g' dX_t + \frac{1}{2} g'' d[X]_t = \left(\mu g' + \frac{1}{2} \sigma^2 g'' \right) dt + \sigma g' dW_t. \quad (3)$$

Note that $0 < \varkappa < 1$ was arbitrary. Therefore, if \varkappa is very close to 0, then g is close to the identity. However, we want to stress that there is no need to make \varkappa particularly small. On the contrary, in general smaller \varkappa gives rise to bigger Lipschitz constants of the transformed coefficients. \varkappa only needs to be smaller than 1 to ensure that g is strictly increasing. Note further that around each discontinuity we build two such splines – one on the right hand side, and one on the left hand side of the discontinuity.

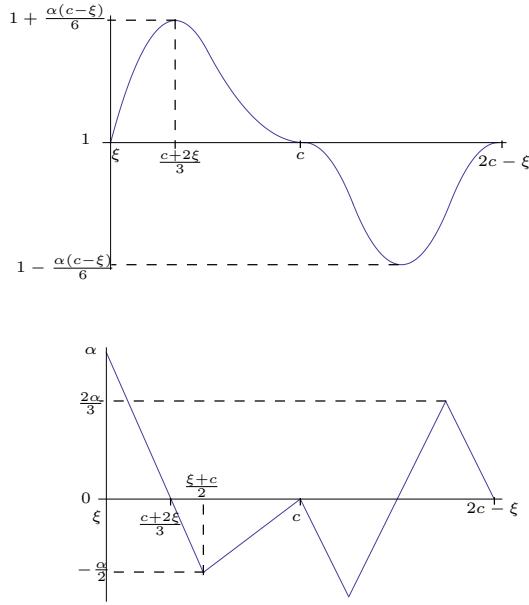


Figure 1: The functions g' and g'' close to ξ .

This is done to reduce the Lipschitz constant of the coefficients of the transformed equation compared to a one-sided compensation of the jumps.

In Section 4 we plot the coefficients of the transformed SDE for Example 4.2.

Now we are ready to prove the existence and uniqueness result, which is obtained by applying the transformation g .

Theorem 2.5. *Under Assumption 2.3 we have that for every $x \in \mathbb{R}$ there exists a unique global strong solution Z to the SDE*

$$\begin{aligned} dZ_t &= \tilde{\mu}(Z_t)dt + \tilde{\sigma}(Z_t)dW_t, \\ Z_0 &= g(x), \end{aligned}$$

where $\tilde{\mu}(z) := \mu(h(z))g'(h(z)) + \frac{1}{2}\sigma^2(h(z))g''(h(z))$ and $\tilde{\sigma}(z) := \sigma(h(z))g'(h(z))$. Furthermore, $h(Z)$ is a unique global strong solution to (2).

For the proof we need the following elementary lemma:

Lemma 2.6. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be piecewise Lipschitz and continuous.*

Then f is Lipschitz on \mathbb{R} .

Proof. There exist $a_1 < \dots < a_m$ such that f is Lipschitz on each of the open intervals $(-\infty, a_1), (a_1, a_2), \dots, (a_m, \infty)$. From the continuity of f we conclude that f is Lipschitz on each of the closed intervals $(-\infty, a_1], [a_1, a_2], \dots, [a_m, \infty)$.

Let L_0, \dots, L_m denote the respective Lipschitz constants and let $L = \max(L_0, \dots, L_m)$. Now let $x, y \in \mathbb{R}$. W.l.o.g. $y < x$. If x and y are in the same interval, then it is obvious that $|f(x) - f(y)| \leq$

$L|x - y|$. Otherwise, we have $y \leq a_k < \dots < a_j \leq x$ with k chosen minimal and j chosen maximal.

$$\begin{aligned}
|f(x) - f(y)| &= |f(x) - f(a_j) + f(a_j) - f(a_{j-1}) + \dots + f(a_k) - f(y)| \\
&\leq |f(x) - f(a_j)| + |f(a_j) - f(a_{j-1})| + \dots + |f(a_k) - f(y)| \\
&\leq L|x - a_j| + L|a_j - a_{j-1}| + \dots + L|a_k - y| \\
&= L(x - a_j) + L(a_j - a_{j-1}) + \dots + L(a_k - y) \\
&= L(x - y) = L|x - y|.
\end{aligned}$$

Thus the assertion is proven. \square

Proof of Theorem 2.5. Define $\hat{\mu}(x) := \mu(x)g'(x) + \frac{1}{2}\sigma^2(x)g''(x)$ and $\hat{\sigma}(x) := \sigma(x)g'(x)$.

Due to Assumption 2.3 (ii), σ is globally Lipschitz. Furthermore, g' is differentiable with bounded derivative, which implies Lipschitz continuity. So σ and g' are both Lipschitz and bounded, thus $\hat{\sigma} = \sigma g'$ is Lipschitz.

To show that $\hat{\mu}$ is Lipschitz, we first note that g is chosen in a way such that $\hat{\mu}$ is continuous. We observe that for $|x|$ large it holds that $g'(x) = 1$ and $g''(x) = 0$. Thus there exists $a > \max(|\xi_1|, |\xi_m|)$ such that $\hat{\mu}$ is Lipschitz on $(-\infty, -a), (a, \infty)$.

Furthermore, $\hat{\mu}$ is Lipschitz on the intervals $(-a, \xi_1), (\xi_1, \xi_2), \dots, (\xi_{m-1}, \xi_m), (\xi_m, a)$ as a sum of products of bounded Lipschitz functions. That means that $\hat{\mu}$ is piecewise Lipschitz.

Hence $\hat{\mu}$ is Lipschitz by Lemma 2.6.

Since g' is bounded away from 0, h' is bounded and thus h is Lipschitz. Thus, also $\tilde{\mu}, \tilde{\sigma}$ are Lipschitz.

From the Lipschitz continuity of $\tilde{\mu}$ and $\tilde{\sigma}$ we get existence and uniqueness of a global strong solution to (3) from [12, Theorem 3.1]. Furthermore, note that Itô's formula holds for h by [8, Problem 7.3]. By applying Itô's formula to h we get that there exists a unique global strong solution to (2). \square

3 Numerical scheme

The numerical scheme is based on the transformation g introduced in Section 2. In fact, we transform the initial value by applying g and then solve the transformed SDE by applying the Euler-Maruyama method. This converges strongly with order 1/2 to the solution of the transformed SDE. Then we apply h to get the solution of our original SDE.

For any $t \geq 0$ and $\delta > 0$ we denote the n -step Euler-Maruyama scheme with step-size δ recursively by

$$\begin{aligned}
\phi^1(z, t, \delta) &:= z + \tilde{\mu}(z)\delta + \tilde{\sigma}(z)(W_{t+\delta} - W_t) \\
\phi^{n+1}(z, t, \delta) &:= \phi^n(\phi^1(z, t, \delta), t + \delta, \delta).
\end{aligned}$$

The scheme for solving (2) looks as follows:

$$\Phi(x, t, T, n) := h(\phi^n(g(X_t), t, T/n)). \quad (4)$$

Now we are ready to prove convergence of scheme (4).

Theorem 3.1. *Let Assumptions 2.3 hold.*

Then scheme (4) converges with order $\gamma = 1/2$ to the solution of (2), i.e.,

$$\mathbb{E}(\|X_T - \Phi(x, 0, T, n)\|^2)^{1/2} \leq C\delta^\gamma.$$

Proof.

$$\begin{aligned}\mathbb{E} (\|X_T - \Phi(x, 0, T, n)\|^2)^{1/2} &= \mathbb{E} (\|h(g(X_T)) - h(\phi^n(g(x), 0, T/n))\|^2)^{1/2} \\ &\leq L_h \mathbb{E} (\|Z_T - \phi^n(g(x), 0, T/n)\|^2)^{1/2},\end{aligned}$$

where Z_T is the exact solution to the transformed SDE. From [9, Theorem 10.2.2] and as $\tilde{\mu}$ and $\tilde{\sigma}$ are Lipschitz we have

$$\mathbb{E} (\|Z_T - \phi^n(z, 0, \delta)\|^2) \leq \tilde{C} \delta.$$

Altogether this yields

$$\mathbb{E} (\|X_T - \Phi(x, 0, T, n)\|^2)^{1/2} \leq C \delta^{1/2},$$

where $C = L_h \sqrt{\tilde{C}}$, and $\delta = T/n$. □

4 Examples

In this section we present numerical examples, where we show the exact choice of the transformation and the transformed parameters. Furthermore, we investigate the convergence of our method and compare it to the Euler-Maruyama method.

Example 4.1. First, we consider the process X mentioned in the beginning and satisfying the equation

$$dX_t = -\text{sign}(X_t)dt + dW_t.$$

That is, $\mu(x) = -\text{sign}(x)$ and $\sigma(x) \equiv 1$. We choose different values of $\varkappa = 1/16, 1/64, 1/256$. Figure 2 shows the estimated \mathcal{L}^2 -error between two consecutive discretizations of the transformed Euler-Maruyama method (EMT 1/ \varkappa) in comparison to crude Euler-Maruyama (EM). This means we calculate $\log \sqrt{\hat{E} \left((X_T^{(k)} - X_t^{(k-1)})^2 \right)}$ and plot it over $\log \delta^{(k)}$, where $X_T^{(k)}$ is the numerical approximation with stepsize $\delta = \delta^{(k)}$ and \hat{E} is an estimator of the mean value using 1024 paths.

We see that for this parameter choice the crude Euler-Maruyama method is even better than the transformed method. Furthermore, we see that the estimated \mathcal{L}^2 -error for the crude Euler-Maruyama method seems to be of order 1. The reason for this is probably that in case of Lipschitz coefficients Euler-Maruyama converges with strong order 1, if the diffusion parameter is constant. However, we cannot expect strong order 1 for the transformed method, since $\tilde{\sigma}$ is not constant, even if σ is constant.

In the next example μ has more than one discontinuity. In addition to the estimated \mathcal{L}^2 -error between two consecutive discretizations we will show how the transformation looks like.

Example 4.2. Now, we consider the SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t.$$

The coefficients are chosen as follows:

$$\begin{aligned}\mu(x) &= \begin{cases} x - 2, & x < -1 \\ 2, & -1 \leq x < -0.5 \\ 1 - x^2, & -0.5 \leq x < 0 \\ x^2, & 0 \leq x < 1 \\ -x - 1, & x \geq 1, \end{cases} \\ \sigma(x) &= \frac{1}{2} \left(1 + \frac{1}{x^2 + 1} \right).\end{aligned}$$

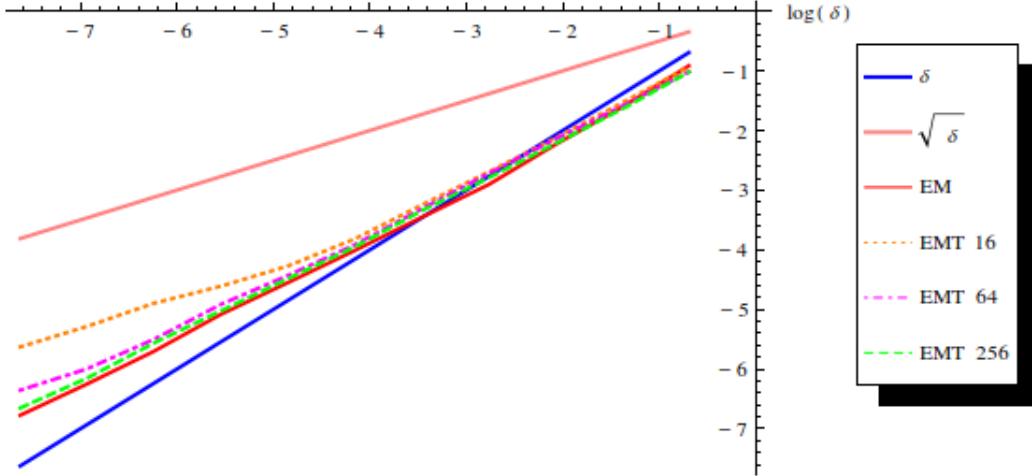


Figure 2: The estimated \mathcal{L}^2 -error for different values of \varkappa for Example 4.1.

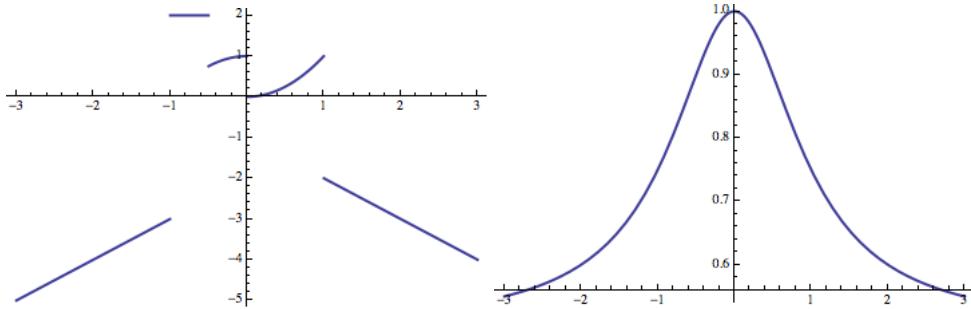


Figure 3: The functions μ and σ .

Figure 3 shows the parameters μ, σ , Figure 4 shows the derivatives of the transformation g , and Figure 5 shows the parameters $\tilde{\mu}, \tilde{\sigma}$ of the transformed SDE.

Again, we choose different values of $\varkappa = 1/16, 1/64, 1/256$. Figure 6 shows the estimated \mathcal{L}^2 -error between two consecutive discretizations of the transformed Euler-Maruyama method (EMT $1/\varkappa$) in comparison to crude Euler-Maruyama (EM).

In this example we reach the calculated order 1/2. Whether the transformed method is better than Euler-Maruyama depends on the choice of \varkappa .

As the last example we solve an SDE appearing in insurance mathematics.

Example 4.3 (Threshold dividend strategy). In [1] the authors study the dividend maximization problem from risk theory in a diffusion model. They find that the optimal dividend policy is of threshold type with constant threshold level b . This means that dividends should be paid at the maximum rate whenever the surplus process X of the insurance company exceeds b . Otherwise, no dividends should be paid. Our numerical scheme enables us to simulate the optimally controlled surplus process from [1]:

$$dX_t = (\theta - K 1_{\{x \geq b\}}) dt + \sigma dW_t,$$

where θ is the drift of the uncontrolled process, and K is the maximum dividend rate. We choose $\theta = 1, K = 1.8, \sigma = 1$, and $b \approx 0.895635$ is the optimal threshold level, which can be calculated as in [1].

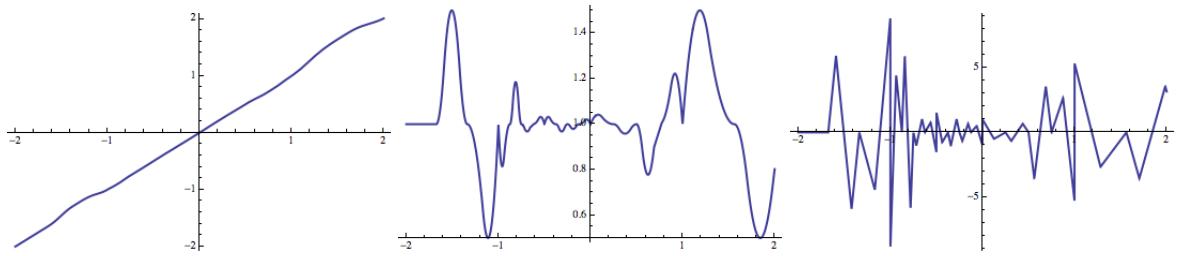


Figure 4: The functions g , g' , and g'' .

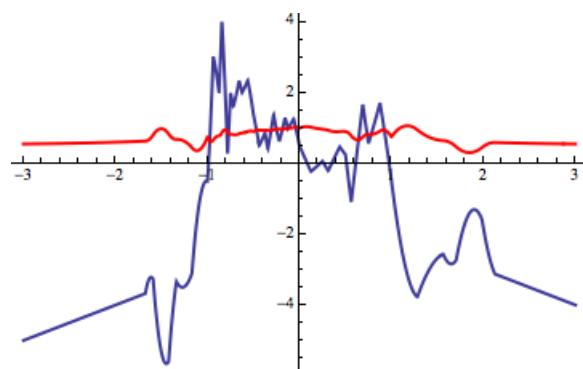


Figure 5: The transformed parameters $\tilde{\mu}$ (blue) and $\tilde{\sigma}$ (red).

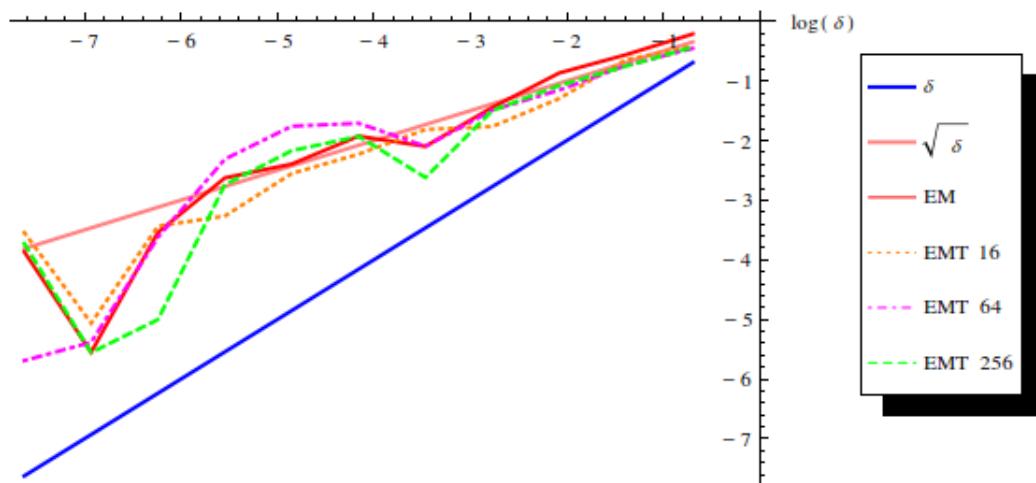


Figure 6: The estimated \mathcal{L}^2 -error for different values of \varkappa for Example 4.2.

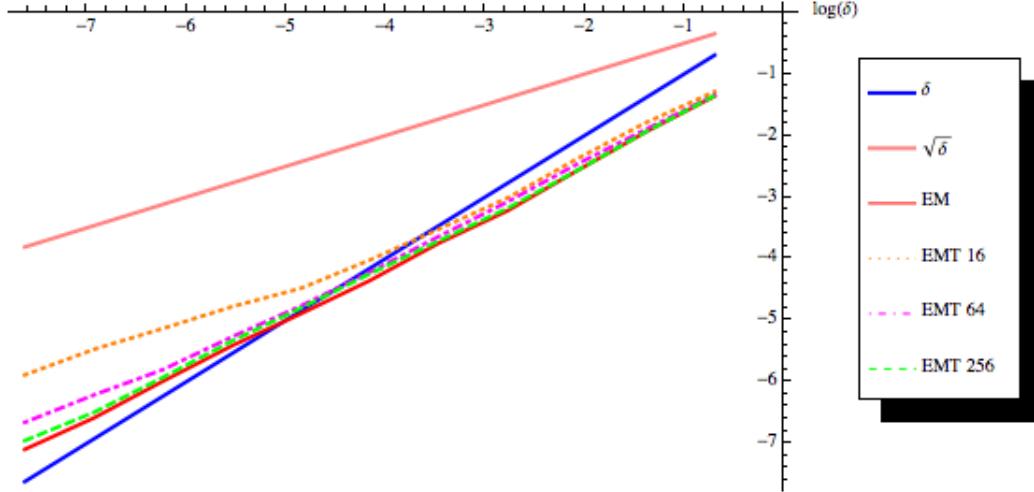


Figure 7: The estimated \mathcal{L}^2 -error for different values of \varkappa for Example 4.3.

Figure 7 shows the estimated \mathcal{L}^2 -error between two consecutive discretizations of the transformed Euler-Maruyama method (EMT $1/\varkappa$) in comparison to crude Euler-Maruyama (EM) for $\varkappa = 1/16, 1/64, 1/256$.

Threshold type control strategies appear frequently when solving stochastic optimization problems in various fields of applied mathematics. Therefore, our scheme potentially serves for a wide range of applications.

References

- [1] S. Asmussen and M. Taksar. Controlled Diffusion Models for Optimal Dividend Pay-Out. *Insurance: Mathematics and Economics*, 20(1):1–15, 1997.
- [2] A. Berkaoui. Euler Scheme for Solutions of Stochastic Differential Equations with Non-Lipschitz Coefficients. *Portugaliae Mathematica*, 61(4):461–478, 2004.
- [3] P. Étoré and M. Martinez. Exact Simulation for Solutions of One-Dimensional Stochastic Differential Equations Involving a Local Time at Zero of the Unknown Process. *Monte Carlo Methods and Applications*, 19(1):41–71, 2013.
- [4] P. Étoré and M. Martinez. Exact Simulation for Solutions of One-Dimensional Stochastic Differential Equations with Discontinuous Drift. *ESAIM: Probability and Statistics*, 2014. Accepted manuscript, arXiv 1301.3019.
- [5] I. Gyöngy. A Note on Euler’s Approximation. *Potential Analysis*, 8:205–216, 1998.
- [6] N. Halidias and P. E. Kloeden. A Note on the Euler–Maruyama Scheme for Stochastic Differential Equations with a Discontinuous Monotone Drift Coefficient. *BIT Numerical Mathematics*, 48(1): 51–59, 2008.
- [7] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden. Strong Convergence of an Explicit Numerical Method for SDEs with Nonglobally Lipschitz Continuous Coefficients. *The Annals of Applied Probability*, 22(4):1611–1641, 2012.

- [8] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1991.
- [9] P. E. Kloeden and E. Platen. *Numerical Solutions of Stochastic Differential Equations*. Stochastic Modelling and Applied Probability. Springer Verlag, Berlin-Heidelberg, 1992.
- [10] A. Kohatsu-Higa, A. Lejay, and K. Yasuda. Weak Approximation Errors for Stochastic Differential Equations with Non-Regular Drift. 2013. Preprint, Inria, hal-00840211.
- [11] G. Leobacher, M. Szölgyenyi, and S. Thonhauser. On the Existence of Solutions of a Class of SDEs with Discontinuous Drift and Singular Diffusion. *Electronic Communications in Probability*, 20(6):1–14, 2015.
- [12] X. Mao. *Stochastic Differential Equations and Applications*. Horwood Publishing Limited, New Delhi, second edition, 2007.
- [13] A. YU. Veretennikov. On Strong Solutions and Explicit Formulas for Solutions of Stochastic Integral Equations. *Mathematics of the USSR Sbornik*, 39(3):387–403, 1981.
- [14] A. YU. Veretennikov. On Stochastic Equations with Degenerate Diffusion with Respect to Some of the Variables. *Mathematics of the USSR Izvestiya*, 22(1):173–180, 1984.
- [15] A. K. Zvonkin. A Transformation of the Phase Space of a Diffusion Process that Removes the Drift. *Mathematics of the USSR Sbornik*, 22(129):129–149, 1974.